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Poisson groups and Schrödinger equation on the circle

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Abstract
We combine differential Galois theory with the theory of Poisson Lie groups to construct a natural Poisson structure on the space of wavefunctions of Schrödinger operators on the circle (at the zero energy level). Applications to KdV-like nonlinear equations are discussed.

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It is well known that the space $\mathcal{H}$ of Schrödinger operators on the circle

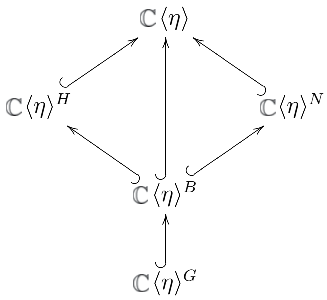
$$H = -\partial_x^2 - u, \qquad u \in C^\infty(S^1), \qquad S^1 \simeq \mathbb{R}/2\pi\mathbb{Z},$$

may be regarded as the phase space for the Korteweg–de Vries hierarchy (with periodic boundary conditions). Along with the KdV equation there exist quite a few closely related equations of the form $u_t = u_{xxx} + F(u, u_x, u_{xx})$ [SS]; the most famous of these 'equations of the KdV type' is the modified KdV equation $v_t = v_{xxx} - 6v^2 v_x$ which is related to the standard KdV by the Miura map $u = v' - v^2$. At least some part of these equations can be understood in the framework of a beautiful group theory picture proposed by Wilson [W]. His idea was to extend the KdV hierarchy to the space of wavefunctions, i.e. solutions of the Schrödinger equation (at zero energy). Let us note that the natural 'algebra of observables' associated with the KdV equation consists of local functionals of the form

$$F[u] = \int_0^{2\pi} f\left(u, \partial_x u, \partial_x^2 u, \ldots\right) \mathrm{d}x,$$

where f is a polynomial (or, more generally, a rational) function of u and of its derivatives. We can identify the observable $F[u]$ and the corresponding density; in other words, our basic algebra of observables is identified with the differential field $\mathbb{C}\langle u\rangle$. According to the elementary theory, for a given u the space $V = V_u$ of solutions of the Schrödinger equation is two-dimensional and for any ϕ, ψ their Wronskian $W = \phi\psi' - \phi'\psi$ is constant. We write (ϕ, ψ) as a row vector; the group $G = SL(2)$ acts naturally on V by right multiplications. We can associate with the space of solutions a bigger differential field $\mathbb{C}\langle\phi, \psi\rangle$. Clearly,

$\mathbb{C}\langle\phi,\psi\rangle \supset \mathbb{C}\langle u\rangle$; as a matter of fact, $\mathbb{C}\langle u\rangle$ is isomorphic to the differential subfield of G-invariants and hence $\mathbb{C}\langle\phi,\psi\rangle \supset \mathbb{C}\langle u\rangle$ is a differential Galois extension with differential Galois group $G = SL(2)$ (we shall speak below simply of Galois groups and Galois extensions, for short). Various subgroups of G give rise to intermediate differential fields. In particular, for $Z = \{\pm I\}$ the associated subfield of invariants is naturally isomorphic to $\mathbb{C}\langle\eta\rangle$, where $\eta = \phi/\psi$; since Z is the center of G, the extension $\mathbb{C}\langle\eta\rangle \supset \mathbb{C}\langle u\rangle$ is again a Galois extension with Galois group $PSL(2) = SL(2)/Z$. Let $B = HN \subset G$ be the standard Borel subgroup consisting of upper triangular matrices and N, H its subgroups of strictly upper triangular and diagonal matrices. They give rise to the following tower of differential extensions:



(1)

The subfields of invariants in this tower admit a simple description: we have $\mathbb{C}\langle\eta\rangle^N \simeq \mathbb{C}\langle\theta\rangle$, where $\theta := \eta'$; in a similar way, the subalgebra of B-invariants is generated by $v := \frac{1}{2}\eta''/\eta' = \frac{1}{2}\theta'/\theta$, the subalgebra of H-invariants is generated by $\rho := \eta'/\eta$ and the subalgebra of G-invariants is generated by $u = v' - v^2$; moreover, we have $u = \frac{1}{2}S(\eta)$, where S is the Schwarzian derivative,

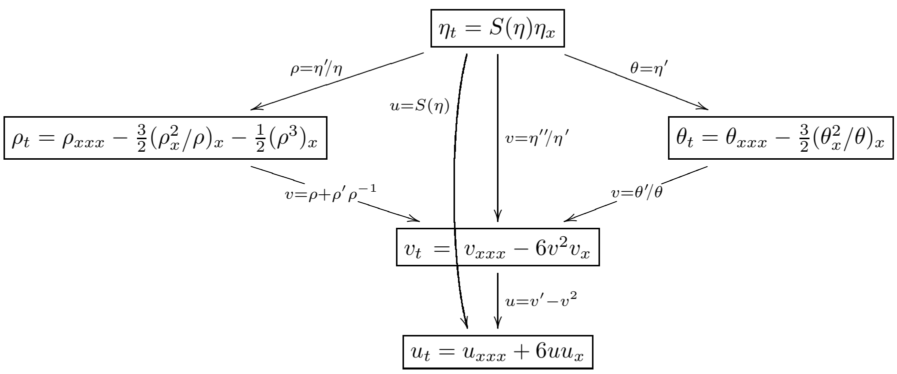
$$S(\eta) = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2.$$

Recall that the crucial property of the Schwarzian derivative is its invariance under projective transformations

$$\eta \mapsto \frac{a\eta + c}{b\eta + d}$$

induced by the right action of G in the space of wavefunctions.

A natural family of evolution equations of the KdV type associated with the tower (1) is represented by the following diagram:



(2)

Until now we did not mention the Hamiltonian structure which is a very important ingredient of the whole picture. Recall that the space of Schrödinger operators on the circle carries a family of natural Poisson structures. We shall be concerned with the so-called second Poisson structure for the KdV equation associated with the third-order differential operator

$$l = \tfrac{1}{2}\partial_x^3 + u\partial_x + \partial_x u. \tag{3}$$

This Poisson structure may be regarded as the Lie–Poisson bracket associated with the Virasoro algebra and arises as a result of the identification of the space $\mathcal{H}$ of Schrödinger operators with (a hyperplane in) the dual space of the Virasoro algebra. The natural idea is to extend in some way this Poisson structure or its substitute to the space of wavefunctions as well as to all intermediate spaces. Wilson's point of view is to look at the symplectic form, because it may be naturally pulled back (at the expense of becoming degenerate, see [W]). A closer look at the situation reveals a difficulty: the relevant 'variational' 2-form is an integral of a density whose differential is not identically zero; rather it is a closed form on the circle and hence its contribution disappears only if we may discard 'total derivatives'. This convention, adopted in formal variational calculus, greatly simplifies many formulae, but sometimes hides important 'obstruction terms'. In Wilson's paper this difficulty is avoided by the tacit assumption that the monodromy matrix is equal to 1. Without this assumption the degenerate 2-forms discussed in his paper are not closed; thus the situation is intrinsically close to the quasi-Hamiltonian formalism of Alekseev, Malkin and Meinrenken [AMM]. An alternative approach, followed in the present paper, is to look at the Poisson structure. Of course, Poisson brackets cannot be pulled back, and hence we have to *guess* a Poisson structure on the extended algebra and then check its consistency with the original bracket. Our main object is the space of wavefunctions and its projectivization associated with the quotient $\eta = \phi/\psi$. For a given potential there is a pair of wavefunctions (ϕ, ψ) with unit Wronskian $W(\phi, \psi) = \phi\psi' - \phi'\psi$ satisfying the quasiperiodicity condition

$$(\phi(x + 2\pi), \psi(x + 2\pi)) = (\phi(x), \psi(x))M, \tag{4}$$

where $M \in SL(2, \mathbb{R})$ is the monodromy matrix. Conversely, any pair of quasiperiodic functions with unit Wronskian satisfies a Schrödinger equation with periodic potential. Let us denote by $\mathcal{V}$ the set of such pairs. For our present purpose it is natural to start with the much bigger space $\mathcal{W}$ of *all* quasiperiodic plane curves,

$$\mathcal{W} = \{(w = (\phi, \psi), M) | w(x + 2\pi) = w(x)M, M \in SL(2, \mathbb{R})\}.$$

The space $\mathcal{W}$ contains the set $\mathcal{W}'$ of all plane curves with nonzero Wronskian as an open subset. Let $\mathcal{C} := C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^\times)$ be the *scaling group* which acts on $\mathcal{W}$ via

$$f \cdot (w, M) = (fw, M). \tag{5}$$

Clearly, $\mathcal{C}$ acts freely on $\mathcal{W}'$ and the quotient may be identified with $\mathcal{V}$. The action of the linear group $G = SL(2)$ on $\mathcal{W}$ is via $g\colon w \mapsto w \cdot g$, $M \mapsto g^{-1}Mg$. Our strategy will be first to equip $\mathcal{W}$ with a Poisson structure and then to use the Wronskian constraint $W(\phi, \psi) = 1$ in order to pass to the space of wavefunctions. We regard the Wronskian constraint as the canonical cross-section of the action of the scaling group; it is very important that the space $\mathcal{V}$ of wavefunctions is realized both as a quotient space with respect to the action of the scaling group and as a submanifold in $\mathcal{W}$.

The key condition which restricts the choice of the Poisson structure on $\mathcal{W}$ is its *covariance* with respect to the group action. This condition puts us in the framework of Poisson group theory, as it allows both $\mathcal{C}$ and G to carry nontrivial Poisson structures, although it does not presume any *a priori* choice of these structures. Let us recall that a Poisson group H is a Lie group equipped with a Poisson structure such that multiplication

$$m : H \times H \to H$$

is a Poisson mapping. A Poisson structure on a manifold $\mathcal{M}$ is *H-covariant* if the natural map $H \times \mathcal{M} \to \mathcal{M}$ which defines an H-action on $\mathcal{M}$ is a Poisson mapping. As it happens, the covariance condition with respect to the action of $\mathcal{C}$ and G on $\mathcal{W}$ together with the natural constraint on the Wronskian make the choice of all relevant Poisson structures almost completely canonical. (In particular, the Poisson bracket on G is fixed up to scaling and conjugation; it is of the standard 'quastriangular' type and the case of zero bracket is excluded.) The Poisson structure on $\mathcal{W}$ constructed in this way is closely related to the so-called *exchange algebras* discovered in the end of 1980s [B]. The point of view adopted in the present paper provides a useful and nontrivial complement to these old results in making explicit the hidden Poisson group aspects of differential Galois theory. It provides a natural route to the usual Virasoro algebra and also to its discrete analog as discussed in [FT, V, FRS].

Lemma 1. *Assume that the Poisson bracket on $\mathcal{W}$ is covariant with respect to the action of $\mathcal{C}$. Then the Poisson structure on $\mathcal{C}$ is trivial and, writing $w = (\phi, \psi)$, the bracket of evaluation functionals has the form*

$$\{w_1(x), w_2(y)\} = w_1(x)w_2(y)R(x, y), \tag{6}$$

where we use the standard tensor notation and write the tensor product $w_1(x)w_2(y)$ as a row vector of length 4; the 'exchange matrix' $R(x, y) \in \mathrm{Mat}(4)$ is given by

$$R(x, y) = \begin{pmatrix} A(x-y) & 0 & 0 & 0 \\ 0 & B(x-y) & -C(y-x) & 0 \\ 0 & C(x-y) & -B(y-x) & 0 \\ 0 & 0 & 0 & D(x-y) \end{pmatrix}.$$

It is convenient to drop temporarily the Jacobi identity condition and to consider all (generalized) Poisson brackets which are covariant with respect to the Galois group action.

Lemma 2. *Let us assume that the Poisson bracket (6) is right-G-invariant (and, consequently, G carries zero bracket); then the exchange matrix has the structure*

$$R_0(x, y) = a(x-y)I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c(x-y) & -c(x-y) & 0 \\ 0 & c(x-y) & -c(x-y) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

where a and c are arbitrary odd functions.

In a more general way, we may now assume that G carries a *nontrivial* Poisson structure. It is well known that such a structure is defined by a *classical r-matrix* $r \in \mathfrak{g} \wedge \mathfrak{g}$; in tensor notation the Poisson bracket of matrix coefficients of $g \in G$ is given by the Sklyanin formula

$$\{g_1, g_2\} = [r, g_1 g_2].$$

Let us choose an arbitrary r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ and equip G with the corresponding Sklyanin bracket.

Lemma 3. *Let us assume that the Poisson bracket (6) is right-G-covariant; then the exchange matrix has the structure*

$$R_r(x, y) = R_0(x, y) + r, \tag{8}$$

where we write $r \in \mathfrak{g} \wedge \mathfrak{g} \subset \mathrm{Mat}(2) \otimes \mathrm{Mat}(2)$ as a 4×4-matrix in the standard way.

For $\mathfrak{g} = \mathfrak{sl}(2)$ the classical Yang–Baxter equation does not impose any restrictions on the choice of r; indeed, it amounts to the requirement that the *Schouten bracket* [BD] $[[r, r]] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ should be ad$\mathfrak{g}$-invariant, but for $\mathfrak{g} = \mathfrak{sl}(2)$ we have $\wedge^3\mathfrak{g} \simeq \mathbb{R}$. Still, we must distinguish two cases:

1. $[[r, r]] = 0$, which happens when $r = 0$ or r is *triangular*;
2. $[[r, r]] = -\epsilon^2 \neq 0$, which happens when r is *quasitriangular*.

Since R_r in (8) is the sum of two terms, the Schouten bracket $[[r, r]]$ gives an extra term to the Jacobi identity for the corresponding exchange bracket.

Lemma 4. *The exchange bracket (6) with exchange matrix (8) satisfies the Jacobi identity if and only if*

$$c(x-y)c(y-z) + c(y-z)c(z-x) + c(z-x)c(x-y) = 0 \tag{9}$$

in case (1) and

$$c(x-y)c(y-z) + c(y-z)c(z-x) + c(z-x)c(x-y) = -\epsilon^2 \tag{10}$$

in case (2).

Functional equation (10) is a version of the so-called *Rota–Baxter equation.* To solve it, one can put $c(x) = \epsilon C(x)$ and express C as a Cayley transform,

$$C(x) = \frac{f(x)+1}{f(x)-1};$$

then (10) immediately yields for f the standard 2-cocycle relation

$$f(x-y)f(y-z)f(z-x) = 1.$$

The obvious solution is thus $C_\lambda(x-y) = \coth\lambda(x-y)$, where λ is a parameter. Setting $\lambda \to \infty$, we obtain a particular solution $C(x-y) = \operatorname{sign}(x-y)$. We shall see that this special solution is the only one which is compatible with the constraint $W = 1$. The solution of the degenerate equation (9) is $c(x) = 1/x$.

So far, the most general Poisson structure on $\mathcal{W}$ still depends on an arbitrary 'structure function' a and on a free parameter. It is easy to check that the Poisson brackets for the ratio $\eta = \phi/\psi$ do not depend on a.

Proposition 5. *We have*

$$\{\eta(x), \eta(y)\} = \epsilon(\eta(x)^2 - \eta(y)^2) - c(x-y)(\eta(x)-\eta(y))^2. \tag{11}$$

Formula (11) defines a family of G-covariant Poisson brackets on the space of 'projective curves' η. In order to establish a connection between these brackets and Schrödinger operators we must take into account the Wronskian constraint which restricts the choice of c. The second structure function a drops out after projectivization and is not restricted by the Jacobi identity. We shall see, however, that the Wronskian constraint suggests a natural way to choose a as well.

Our next proposition describes the basic Poisson bracket relations for the Wronskian.

Proposition 6. *We have*

$$\{W(x), \phi(y)\} = (c(x-y) - 2a(x,y))W(x)\phi(y) - c'(x-y)\phi(x)[\phi(x)\psi(y) - \psi(x)\phi(y)]. \tag{12}$$

By symmetry, a similar formula holds for $\{W(x), \psi(y)\}$.

Formula (12) immediately leads to the following crucial observation:

Proposition 7. *The constraint* $W = 1$ *is compatible with the Poisson brackets for the scaling invariant* η *if and only if the last term in (12) is identically zero; this is possible if and only if* $c'(x-y)$ *is a multiple of* $\delta(x-y)$*, i.e., if* $c(x-y)$ *is a multiple of* $\mathrm{sign}(x-y)$.

It is important that the Wronskian constraint excludes the possibility that $\epsilon = 0$ and hence the corresponding Poisson structure on G must be quasitriangular (case (2)). From now on, without restricting the generality, we can fix $\epsilon = 1$ and assume that r is the *standard quasitriangular r-matrix*, $r = e \wedge f$. Other possible choices differ by rescaling and conjugation; it is important to note that our particular choice represents a kind of peculiar symmetry breaking, since it completely fixes the basis in $\mathbb{R}^2$, as well as the choice of the Borel subgroup $B = HN \subset G$. Note that with this choice B, H, N, together with the opposite Borel subgroup, give the complete list of *admissible subgroups* of G. (A subgroup of a Poisson group is called admissible if its invariants in the Poisson algebra of functions are closed with respect to the Poisson bracket.) We arrive at the important conclusion: *differential Galois group spontaneously becomes a Poisson group and the choice of its Poisson structure is essentially unique.*

Proposition 8. *Let us assume that* $c(x-y) = \mathrm{sign}(x-y)$*; then the Poisson bracket relations for the Wronskian are given by*

$$\{W(x), W(y)\} = (\mathrm{sign}(x-y) - 2a(x-y))W(x)W(y), \tag{13}$$

or, equivalently

$$\{\log W(x), \log W(y)\} = (\mathrm{sign}(x-y) - 2a(x-y)). \tag{14}$$

Formulae (12) and (13) suggest the following distinguished choice of a:

Proposition 9. *Assume that a is so chosen that*

$$\mathrm{sign}(x-y) - 2a(x-y) = \delta'(x-y).$$

(In other words, $a(x-y)$ *is the distribution kernel of the operator* $\frac{1}{2}(\partial^{-1} - \partial)$*.) Then, (i) the logarithms of Wronskians form a Heisenberg Lie algebra, the central extension of the Abelian Lie algebra of* $\mathcal{C}$*; (ii) let* $\mathcal{C}' = \mathcal{C}/\mathbb{C}^*$ *be the quotient of the scaling group over the subgroup of constants;* $\log W$ *is the moment map for the action of* $\mathcal{C}'$ *on* $\mathcal{W}$.

Recall that according to the general theory the Poisson bracket relations for the moment map may reproduce the commutation relations for a central extension of the original Lie algebra. This is precisely what happens in the present case.

With this choice of a and c the Poisson geometry of the space $\mathcal{V}$ of wavefunctions becomes finally quite transparent: $\mathcal{V}$ arises as a result of Hamiltonian reduction with respect to $\mathcal{C}$ over the zero level of the associated moment map. The constraint set $\log W = 0$ is (almost) non-degenerate (i.e. this is a 2nd class constraint, according to Dirac). The projective invariants $\eta(x)$ commute with the Wronskian and hence their Poisson brackets are not affected by the constraint.

The description of the Poisson structure on $\mathcal{V}$ is completed by the Poisson brackets for the monodromy.

Proposition 10. *The Poisson covariant brackets for the monodromy have the form*

$$\begin{aligned} \{w(x)_1, M_2\} &= w(x)_1(M_2 r_+ - r_- M_2), \\ \{M_1, M_2\} &= M_1 M_2 r + r M_1 M_2 - M_2 r_+ M_1 - M_1 r_- M_2. \end{aligned} \tag{15}$$

The Poisson bracket for the monodromy is precisely the Poisson bracket of the dual Poisson group G^* [S]. In other words, the 'forgetting map' $\mu : (w, M) \mapsto M$ is a Poisson morphism from $\mathcal{W}$ into the dual group G^*. This mapping is of special importance.

Proposition 11. *The mapping μ is the non-Abelian moment map associated with the right action of G on $\mathcal{W}$ (see, e.g. [BB] for the general definition of non-Abelian moment maps associated with Poisson group actions).*

Returning to the extension tower (1), we can now list all Poisson bracket relations in the differential algebra $\mathbb{C}\langle\eta\rangle$ and its various subalgebras which correspond to different admissible subgroups of G.

Proposition 12.

(i) The basic Poisson bracket relations in $\mathbb{C}\langle\eta\rangle$ are given by

$$\{\eta(x), \eta(y)\} = \eta(x)^2 - \eta(y)^2 - \operatorname{sign}(x-y)(\eta(x)-\eta(y))^2. \tag{16}$$

(ii) For $\theta = \eta'$ we have

$$\{\theta(x), \theta(y)\} = 2\operatorname{sign}(x-y)\theta(x)\theta(y). \tag{17}$$

(iii) The Poisson bracket relations for $v = \frac{1}{2}\eta''/\eta' = \frac{1}{2}\theta'/\theta$ are given by

$$\{v(x), v(y)\} = \tfrac{1}{2}\delta'(x-y). \tag{18}$$

(iv) For $u = \frac{1}{2}v' - v^2 = S(\eta)$ we have:

$$\{u(x), u(y)\} = \tfrac{1}{2}\delta'''(x-y) + \delta'(x-y)[u(x)+u(y)]. \tag{19}$$

(v) All arrows in commutative diagram (1) are Poisson morphisms.

Formula (19) reproduces the standard Virasoro algebra; in other words, the Poisson algebra (16) constructed from general covariance principles is indeed an extension of the Poisson–Virasoro algebra.

The Poisson bracket relations (17)–(19) listed above are particularly simple, since their rhs is algebraic. Since the basic Poisson bracket relations (16) are nonlocal, this need not always be the case. This is what happens in the case of H-invariants.

Proposition 13. *(i) The differential subalgebra of H-invariants in $\mathbb{C}\langle\eta\rangle$ is generated by $\rho = \eta'/\eta$. (ii) The Poisson brackets for ρ have the form*

$$\{\rho(x), \rho(y)\} = 2\rho(x)\rho(y)\left[\sinh\int_x^y \rho(s)\,\mathrm{d}s + \operatorname{sign}(x-y)\cosh\int_x^y \rho(s)\,\mathrm{d}s\right].$$

Let us now return to the commutative diagram (2) relating various KdV-like equations on different levels of the extensions tower (1). At the bottom of the tower we have the standard KdV equation which is generated with respect to the Virasoro bracket by the Hamiltonian

$$h[u] = \int u^2\,\mathrm{d}x. \tag{20}$$

This Hamiltonian and all its higher companions are associated with trace identities for the Schrödinger operator and hence are G-invariant. Thus they may be lifted to all levels of the extension tower, and on each level of this tower they generate a system of commuting flows; moreover, all 'differential substitutions' relating various KdV-like equations in diagram (2) are Poisson maps which commute with the flows. The mutual relations of these equations were discussed by Wilson [W] who for the first time pointed out the relevance of the differential Galois approach to this problem; however, the Hamiltonian description which we propose is totally different.

Concluding Remarks

The 'spontaneous symmetry breaking' which leads to the emergence of nontrivial Poisson structures on symmetry groups and eventually of quantum groups has its analogs in various problems related to completely integrable systems. The same ideology applied to second-order difference operators on a lattice gives rise to the so-called lattice Virasoro algebra [FT, FRS] and to its Poisson Galois extension. The closely related case of second-order q-difference operators is more subtle; we shall address both problems in a separate report.

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